Mortality, Time Preference and Life-Cycle Models

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Abstract

I compare two different ways of integrating mortality into life-cycle models: the standard additive model with time preferences, on the one hand, and a formulation that rules out the existence of time preferences, but allows for temporal risk aversion, on the other hand. These models are of similar complexity, but substantially differ in their fundamental assumptions. I show, however, that the latter formulation can reproduce all the predictions of the additive models, as long as life-cycle behaviors under a given mortality pattern are considered. It leads, nonetheless, to radically different predictions for the effects of mortality changes. The impact of mortality on impatience and intertemporal choices may actually be very different from what is usually assumed.

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1 Introduction

Heterogeneity in mortality across periods and regions is one of the most striking features of recent human history. For example, in Sweden, life expectancy at birth rose from about 35 years in 1800 to 80 years in 2000. In Zimbabwe, a country severely hit by the HIV/AIDS epidemics, life expectancy dropped from 60 years in 1985-1990 to 37 in 2003. Clearly, if we want to understand why people living in different times or countries have different life-cycle behaviors, or if we aim at providing insights on the economic impact of mortality changes, we need to think very carefully about how mortality is integrated into life-cycle models.

Surprisingly, there is a dearth of literature on this topic. Most life-cycle models that account for uncertain survival follow Yaari's (1965) seminal paper. According to Yaari, survival uncertainty can be simply incorporated into life-cycle models by weighting the utility derived from future consumption by survival probabilities. Thus, in Yaari's model, expected lifetime utility can be written as:

$$\int_0^{+\infty} s(t)\alpha(t)u(c(t))dt \tag{1}$$

where $c(t)$ is the consumption at age t, $s(t)$ the probability of being alive at age t and $\alpha(t)$ the subjective discount function. Yaari's model has undoubtedly become the model of reference for those interested in the effects of mortality on life-cycle behaviors. A few alternative models have been proposed, as with Moresi (1999). To my knowledge, however, all applied and theoretical studies that have focused on the economic impact of mortality have kept Yaari's linear formulation as the keystone of their analysis[1].

At the origin of Yaari's model is the fundamental assumption that preferences over lotteries involving lives of different length, can be modeled in the standard expected utility framework using a von Neumann-Morgenstern utility function of the form:

$$V(c,T) = \int_0^{T} \alpha(t)u(c(t))dt \tag{2}$$

where $c(t)$ is the consumption at age t and T the length of life[2]. Implicit in this formu-

[1] See for example Barro and Friedman (1977), Ulph and Hemming (1980), Davies (1981), Sheshinski and Weiss, (1981), Abel (1986), Hurd (1989), Leung (1994), Brown (2001), Eckstein and Tsiddon (2004), Gan and Gong (2004).

[2] The above von Neumann-Morgenstern utility function exactly corresponds to the one chosen by

lation are three fundamental assumptions. The first is an assumption of separability: (1) the marginal rate of substitution between the consumption at two different ages is unaffected by the level of consumption at another age and by the length of life. The second and third assumptions are, as noted by Yaari, "independence of age" assumptions. Specifically, it is assumed that (2) the marginal rate of substitution between the length of life and consumption at the end of life is independent of the length of life, and (3) risk aversion with respect to consumption at a given age is independent of age (these concepts will be made clearer in the following section). These assumptions can, of course, be a matter of discussion. They respond, however, to defendable arguments. The second and third assumptions (independence of age) are necessary if we aim at explaining the age-related heterogeneity in behaviors without making *ad hoc* assumptions on heterogeneity in preferences. The first assumption (separability) has clearly been made for simplicity's sake. It will have to be relaxed sooner or later, but that will be at the cost of an increase in complexity[3]. Before going in that direction, it is worth exploring the whole set of preferences that satisfy these three assumptions.

In this paper, I show that Yaari's formulation is only one of the two general formulations that share these three fundamental properties. The other one involves assuming that preferences over consumption profiles and lengths of life are represented by a von Neumann-Morgenstern utility function of the form:

$$W(c,T) = \phi\left(\int_0^T u(c(t))dt\right) \tag{3}$$

This alternative formulation, which emerges naturally from a reading of Kihlstrom and Mirman (1974), has been ignored in the economic literature. I shall argue that such a formulation is definitely worth considering for the following reasons:

1. It is no more complex than the standard additive model suggested by Yaari.

2. It suggests an original theory for human impatience. Individuals have no time

Yaari (see equation (1), page 137, in Yaari, 1965). Most economic papers directly refer to the representation in terms of expected utility (equation (1) in this paper and equation (13), page 142, in Yaari's paper). Both representations are obviously equivalent, but for a theoretical discussion on preferences it proves more insightful to discuss the properties of the von Neumann-Morgenstern utility function than those of the expected utility function (in which preferences and uncertainty are already combined).

[3]This assumption is relaxed in Bommier (2005), but in that paper, an assumption of stationarity is made.

preferences and impatience exclusively results from the combination of risk aversion and uncertainty.

3. It accounts for temporal risk aversion.

4. Separation of risk aversion and intertemporal elasticity of substitution is made possible.

5. As long as we consider life-cycle behaviors under a given (non-degenerate) mortality pattern, it can reproduce (up to infinitesimally small differences) all the predictions of the additive models with non-negative rates of time preference.

6. It leads to predictions on the economic impact of mortality changes that are very different from those obtained with the additive model.

The fifth and sixth points deserve special attention. The fifth point implies that there is no way to prove that Yaari's model is superior to the suggested alternative without considering heterogeneity in mortality across agents. But, to my knowledge, the validity of Yaari's model has never been tested by empirical studies using such heterogeneity. The economic literature has thus focused on a specification that separates impatience and risk aversion, while a no more complex model where impatience results from risk aversion is shown to have at least as much predictive power.

Nevertheless, the sixth point tells that it is crucial to choose between the two models to analyze the impact of mortality changes. Since there is no empirical evidence today that might suggest the superiority of Yaari's model, the robustness of the economic literature that discusses the effects of mortality changes appears to be questionable.

The remainder of the paper is organized in the following way. In the first section, I present an axiomatic construction of preferences. This will lead us to consider two classes of von Neumann-Morgenstern utility functions: the well known additive utility function shown in equation (2), and the non-additive one shown in equation (3). The following sections aim at comparing these two formulations. More precisely, in Sections 3 through 8, I go on to review the six arguments detailed above. Section 9 looks at technical difficulties that appear when working with the non-additive model and suggests ways to deal with them. Section 10 discusses the main conclusions that can be drawn from the present paper.

2 Axiomatic Construction

In this paper, I will view a "life" as being a pair (c, T), where c is an infinitely long consumption profile, and T a (finite) length of life. The set of possible lives will, therefore, be:

$$X = C^{\infty}(\mathbb{R}^{+}, \mathbb{R}^{+}) \times \mathbb{R}^{+}$$

This representation might seem odd at first sight since consumption has not been constrained to zero after death. Instead, consumption after death can theoretically take any non-negative value. However, as I will assume that people do not care for consumption after death, my results will be formally equivalent to what we would obtain if consumption was constrained to zero after death.

I aim to discuss the properties of the preferences that make it possible to rank lotteries whose outcomes are in X. I will, in fact, explore all the preferences that satisfy the following axioms:

Axiom 1 *Individuals' preferences can be represented by a twice continuously differentiable von Neumann-Morgenstern utility function $U(c, T)$ defined over X.*

This axiom implies that, unlike Moresi (1999), Drouhin (2005) or Halevy (2005), we remain within the standard expected utility framework. The smoothness assumption has been added for convenience.

Axiom 2 *Individuals do not care for consumption after death.*

By this axiom, I mean that for any consumption profile, c, any length of life, T, and any age $t > T$:

$$\frac{\partial U(c, T)}{\partial c(t)} = 0$$

where $\frac{\partial U(c,T)}{\partial c(t)}$ is the Volterra derivative of $U(c, T)$ with respect to consumption at time t.[4]

[4] Volterra derivatives make it possible to define, in an intuitive way, the derivative of $U(c, t)$ with respect to consumption at a single moment in time. The reader may refer to Volterra (1913) for the definition of this kind of derivative or to Ryder and Heal (1973) for the use of such a concept in economics. For the remainder of the paper, we simply have to bear in mind that $\frac{\partial U(c,T)}{\partial c(t)} dcdt$ gives the infinitesimal change of $U(c, T)$ when consumption increases of dc during dt periods of time around t.

Axiom 3 *Utility is increasing in consumption occurring before death.*

Formally, I assume that for any c, T and any $t < T$:

$$\frac{\partial U(c,T)}{\partial c(t)} > 0$$

Axiom 4 *The marginal rate of substitution between consumption at two different ages is independent of the length of life and independent of consumption at another age.*

Expressed formally, this statement means that for any c, T and any three distinct ages $t_1, t_2, t_3 \in (0,T)$, we must have:

$$\frac{\partial}{\partial T}\left(\frac{\frac{\partial U(c,T)}{\partial c(t_1)}}{\frac{\partial U(c,T)}{\partial c(t_2)}}\right) = 0 \text{ and } \frac{\partial}{\partial c(t_3)}\left(\frac{\frac{\partial U(c,T)}{\partial c(t_1)}}{\frac{\partial U(c,T)}{\partial c(t_2)}}\right) = 0$$

The last two axioms, given below, aim at formally expressing age independence assumptions that are implicit in Yaari's formulation. In Yaari's additive formulation, instantaneous utility functions, $\alpha(t)u(\cdot)$, are assumed to depend on age only through the scaling factor $\alpha(t)$. However, when we abandon the additive structure, there does not exist a straightforward notion of instantaneous utility. Some effort is then required to give a preference based definition of this age independence assumption. In fact assuming that the instantaneous utility (in the additive specification) is age independent, up to a scaling factor, implies that (1) risk aversion with respect to instantaneous consumption is independent of age (2) the ratio of the marginal utility of life extension over the marginal utility of consumption ($\frac{u}{u'}$ in the additive case) is independent of age. We formally expressed these properties by the following two axioms.

Axiom 5 *Risk aversion with respect to instantaneous consumption is independent of age.*

In other words, for any c, T and any t_1, $t_2 \in (0,T)$, we have:

$$c(t_1) = c(t_2) \Rightarrow \frac{\frac{\partial^2 U(c,T)}{\partial^2 c(t_1)}}{\frac{\partial U(c,T)}{\partial c(t_1)}} = \frac{\frac{\partial^2 U(c,T)}{\partial^2 c(t_2)}}{\frac{\partial U(c,T)}{\partial c(t_2)}}$$

Axiom 6 *For any consumption profile, c, and any length of life T, the marginal rate of substitution between length of life and consumption at the end of life depends on c and T only through $c(T)$.*

This means that for any two consumption profiles c_1, c_2 and any two lengths of life T_1, T_2:

$$c_1(T_1) = c_2(T_2) \Rightarrow \frac{\frac{\partial U(c_1,T_1)}{\partial T_1}}{\frac{\partial U(c_1,T_1)}{\partial c_1(T_1)}} = \frac{\frac{\partial U(c_2,T_2)}{\partial T_2}}{\frac{\partial U(c_2,T_2)}{\partial c_2(T_2)}}$$

These axioms being stated, it is possible to express the following result:

Theorem 1 *Axioms 1-6 are fulfilled if and only if individuals' preferences can be represented by a von Neumann-Morgenstern utility function of the form:*

$$U(c,T) = \phi\left(\int_0^T \alpha(t)u(c(t))dt\right) \tag{4}$$

where α and ϕ are twice continuously differentiable functions such that α, u' and ϕ' are positive and such that $\phi'' = 0$ or/and $\alpha' = 0$ (so that ϕ must be linear or/and α constant).

Proof. See appendix A. ■

Theorem 1 shows that complying with Axioms 1 to 6 leaves us with two possibilities. For convenience, I will give them different names:

Definition 1 *Let us say that individuals have:*

- "additive preferences" (or also Yaari-type preferences) if they are represented by a von Neumann-Morgenstern utility function of the form:

$$U^{add}(c,T) = \int_0^T \alpha(t)u(c(t))dt \tag{5}$$

with $\alpha > 0$ and $u' > 0$.

- "time neutral preferences" if they are represented by a von Neumann-Morgenstern utility function of the form:

$$U^{tn}(c,T) = \phi\left(\int_0^T u(c(t))dt\right) \tag{6}$$

with $\phi' > 0$ and $u' > 0$.

Let us now compare these two types of preferences and develop the six arguments outlined in the introduction.

3 First Argument: On Models' Complexity

Before comparing two models, it is important to evaluate the complexity of each model individually. It is clear that extending models into more complex ones makes them more apt to fit empirical data. However, increasing complexity has an obvious disadvantage, as identification problems grow.

Here, the comparison between the additive and the time neutral model is simple: both models have exactly the same degree of complexity. In both cases, preferences are described by two functions: u and α in the additive case, and u and ϕ' in the time neutral case (adding a constant to ϕ does not affect the preferences).

In fact, the representations of the preferences by the couple of functions u and α or u and ϕ' are not unique, and normalization assumptions must be added in order to obtain uniqueness. In both cases, however, the number and the kind of constraints needed to obtain a unique representation are the same. Indeed, it is straightforward that it can always be assumed that:

$$\begin{array}{lll} \alpha(0) = 1 \quad \text{and} \quad u'(1) = 1 & \text{in the additive case} \\ \phi'(0) = 1 \quad \text{and} \quad u'(1) = 1 & \text{in the time neutral case} \end{array}$$

and that once these constraints are introduced, the representations of preferences provided by equations (5) and (6) are unique.

In fact, both the additive formulation and the time neutral one can be seen as diverging extensions of the simplest case where preferences are represented by:

$$U_0(c,T) = \int_0^T u(c(t))dt$$

with $u' > 0$. Preferences represented by U_0 are both additive and time neutral (Definition 1). As we will see in the following two sections, the additive preferences extend the above formulation by introducing time preferences, while the time neutral preferences introduce temporal risk aversion.

4 Second Argument: On Time Preferences

The concept of pure time preference is an ordinal concept representing impatience in a context without uncertainty. It can be summarized by the rate of time preference, which in the continuous time framework is usually defined as follows:

Definition 2 *For any length of life T, any time $t < T$ and any consumption path c, the rate of time preference is defined by:*

$$\rho(c,t,T) = -\frac{d}{dt}\left(\log(\frac{\partial U(c,T)}{\partial c(t)})\right)|_{\frac{dc(t)}{dt}=0}$$

Note that I use the notation $\rho(c,t,T)$ to stress that, in general, the rate of time preference can depend on c, t and T. However, with the preferences we are considering, the rate of time preference at time t only depends on t. Indeed:

Proposition 1 *In the additive model, the rate of time preference is given by:*

$$\rho^{add}(c,t,T) = \frac{-\alpha'(t)}{\alpha(t)} \tag{7}$$

In the time neutral model, it is given by:

$$\rho^{tn}(c,t,T) = 0 \tag{8}$$

Proof. From (5) we derive $\frac{\partial U^{add}(c,T)}{\partial c(t)} = \alpha(t)u'(c(t))$, which implies (7). From (6) we derive $\frac{\partial U^{tn}(c,T)}{\partial c(t)} = u'(c(t))\phi'\left(\int_0^T u(c(t))dt\right)$, which implies (8). ■

Here, lies a fundamental difference between the two models. In the additive case, people can have pure time preferences while the time neutral model excludes this possibility.

Still, as will be explained in Section 7, agents with time neutral preferences may exhibit any kind of (positive) impatience when confronted with lifetime uncertainty. The additive and time neutral models therefore suggest two very different theories for human impatience. In the standard approach, supported by the additive model, impatience is inherent to human nature and would exist even in the absence of uncertainty. This position was strongly criticized by Pigou (1920) and Ramsey (1928) who respectively considered time preference as “wholly irrational” or arising from “the weakness of imagination", but nonetheless became the dominant view in economic theory afterwards.

On the other hand, the time neutral model, takes for granted that risk aversion and mortality are inherent to human nature. It then suggests that human impatience may exclusively result from a rational response to the risk of death.

The interest of each interpretation might be debated on philosophical grounds. Instead, I will focus on pragmatic matters and show why opting for one or the other interpretation might be crucial for very concrete social issues, and in particular to understanding the impact of mortality changes.

5 Third Argument: On Temporal Risk Aversion

Temporal risk aversion is an adaptation of the general notion of "multivariate risk aversion" of Richard (1975), to the case of intertemporal choice under uncertainty. It is used in Ahn (1989) and Van der Ploeg (1993). To obtain an intuitive notion of what temporal risk aversion is, consider the simple case of an individual who lives over two periods. An individual is temporally risk averse if for any $c_1 < C_1$ and $c_2 < C_2$ he prefers the lottery that gives (c_1, C_2) or (C_1, c_2) with equal probability to the lottery that gives (c_1, c_2) or (C_1, C_2) with equal probability. To quote Richard (1975), a temporally risk averse consumer prefers getting some of the "best" and some the "worst", to taking a chance on all of the "best" or all of the "worst". Richard (1975) shows that temporal risk aversion is related to the cross derivative of the utility function. In continuous time, temporal risk aversion can be defined as follows:

Definition 3 *An individual exhibits:*

- *temporal risk aversion if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} < 0$ *for all* $t_1,\ t_2 < T$ *with* $t_1 \neq t_2$.
- *temporal risk neutrality if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} = 0$ *for all* $t_1,\ t_2 < T$ *with* $t_1 \neq t_2$.
- *temporal risk proneness if* $\frac{\partial^2 U(c,T)}{\partial c(t_1)\partial c(t_2)} > 0$ *for all* $t_1,\ t_2 < T$ *with* $t_1 \neq t_2$.

It is then fairly simple to note that:

Proposition 2 *Agents with additive preferences exhibit temporal risk neutrality. Agents with time neutral preferences exhibit temporal risk aversion if ϕ is concave, temporal risk neutrality if ϕ is linear, and temporal risk proneness if ϕ is convex.*

Proof. In the additive case $\frac{\partial^{add}U(c,T)}{\partial c(t_1)} = \alpha(t_1)u(c(t_1))$ and $\frac{\partial^2 U^{add}(c,T)}{\partial c(t_1)\partial c(t_2)} = 0$.

In the time neutral case $\frac{\partial^{tn}U(c,T)}{\partial c(t_1)} = u'(c(t_1))\phi'\left(\int_0^T u(c(t)dt\right)$ and $\frac{\partial^2 U^{tn}(c,T)}{\partial c(t_1)\partial c(t_2)} = u'(c(t_1))u'(c(t_2))\phi''\left(\int_0^T u(c(t)dt\right)$. ■

This is the second fundamental difference between the two models. The additive model rules out temporal risk aversion while the time neutral model allows for it.

Temporal risk aversion matters when considering attitude towards risks that have durable consequences, since risks of this kind affect individuals in several periods of time. This is for example the case for risks related to wealth investment, since current wealth affects individuals consumption in future periods. This explains why temporal risk aversion plays a central role in Ahn (1989), Van der Ploeg (1993) or Bommier and Rochet (2006) who study optimal saving and portfolio choices in models where the horizon is infinite or known with certainty.

A risk that indisputably has longlasting consequences is that of mortality. Indeed, the risk of dying at time t is nothing other than the risk of being put in the "death state" for all times subsequent to t. Thus, we expect temporal risk aversion to deeply affect rational attitudes towards the risk of death. In fact, there are several issues related to lifetime uncertainty where temporal risk aversion plays a key role. The present paper focuses on the case where mortality is exogenous. In particular, it is shown that the combination of temporal risk aversion with mortality generates impatience, which may radically affect our understanding of the impact of mortality changes. Bommier and Villeneuve (2004) complements this study by looking at endogenous mortality choices.

Given the obvious durability of death, it is intriguing that the economic literature that deals with human mortality focuses on the additive specification which assumes temporal risk neutrality. Several papers, such as Levhari and Mirman (1977) and Eeckhoudt and Hammitt (2004), discuss the role of "risk aversion". But, these papers only consider the additive specification and discuss the role of the curvature of the instantaneous utility function u, which has no impact on temporal risk aversion. We know however from the fundamental contribution of Kihlstrom and Mirman (1974) that, strictly speaking, increasing individuals' risk aversion does not involve changing the curvature of u, but taking a concave transformation of the intertemporal utility function. This is what is done with the time neutral model, where temporal risk aversion naturally arises.

To end this section, let us remark that the curvature of the function ϕ, which generates temporal risk aversion in the time neutral model, can be related to individuals' risk aversion with respect to life duration. Imagine the (fictive) case of individuals who have to choose between lotteries involving a single constant consumption path, but different life durations. Consumption being the same in all outcomes, these individuals only have to rank lotteries on a single dimensional variable: life duration. Their choices are then governed by their risk aversion with respect to life duration which can be measured by a standard Arrow-Pratt coefficient:

$$-\frac{\frac{\partial^2 U(c,t)}{\partial T^2}}{\frac{\partial U(c,t)}{\partial T}}$$

It is a matter of simple calculation to show that in the time neutral model, this coefficient equals $u(c)\frac{-\phi''(Tu(c))}{\phi'(Tu(c))}$. Considering such simple lotteries may therefore help to understand the economic meaning of assumptions that might be made about ϕ. For example, assuming that ϕ is concave would involve assuming positive risk aversion with respect to length of life, while assuming that $-\frac{\phi''}{\phi'}$ is decreasing would involve assuming decreasing risk aversion with respect to length of life.

6 Fourth Argument: On Risk Aversion and Intertemporal Elasticity of Substitution

A standard argument against the additive model bears on its inability to separate risk aversion and intertemporal elasticity of substitution. The time neutral model provides a simple way to achieve this separation. The function ϕ that enters into the definition of the time neutral utility function has indeed no effect on the intertemporal elasticity of substitution (ordinal preferences do not depend of the function ϕ) while it does affect relative risk aversion.

The fact that considering monotonic transformations of an additive separable utility function makes it possible to separate intertemporal substitutability and risk aversion has been known for years (at least since Kihlstrom and Mirman, 1974). Such a separation between risk aversion and intertemporal elasticity of substitution has been used to explain the equity premium puzzle (Ahn, 1989), or to model precautionary saving (van der Ploeg, 1993). This approach has not become very popular however, as it generates unappealing properties when there are pure time preferences (see the discussion in Epstein and Zin, 1989, p 950-952). As long as the existence of pure time preferences were not questioned, the best option to separate intertemporal substitutability and risk aversion appeared to leave the expected utility framework, as in Epstein and Zin (1989) or Weil (1990). Still, the expected utility framework remains a suitable option when ruling out the existence of time preferences, as in the time neutral model.

7 Fifth Argument: On Life-Cycle Behavior Under an Exogenous Mortality Pattern

In this section, I consider the case where individuals face an exogenous mortality pattern. Throughout the section, mortality will be described either by the distribution of the age at death $d(t)$, by the survival function $s(t) = 1 - \int_0^t d(\tau)d\tau$ or by the hazard rate of death $\mu(t) = -\frac{s'(t)}{s(t)} = \frac{d(t)}{s(t)}$. Even though, in this section, I do not compare what is obtained with different mortality patterns (this is the purpose of Section 8), I will introduce an index μ whenever I want to stress that an object depends on the mortality pattern.

Rational individuals with a von Neumann-Morgenstern utility function $U(c, T)$ who face this exogenous mortality pattern have preferences on consumption profiles given by the following expected utility:

$$E_\mu U(c) \equiv \int_0^{+\infty} d(T)U(c, T)dT \tag{9}$$

A crucial point is that although the time neutral representation assumes that people have no pure time preferences, temporal risk aversion, together with uncertainty on the length of life, generate non-trivial time discounting. The intuition, stressed in Bommier (2006), is that if people cannot avoid the risk of dying young, they should prefer consuming early in life in order to avoid the very low level of lifetime utility, which would result from simultaneously having a short life and low levels of instantaneous consumption. This intuition can be formalized by looking at the rate of discount at time t.

Definition 4 *For any consumption profile c, the rate of discount at time t is defined by:*

$$RD_\mu(c,t) = -\frac{d}{dt}(\log(\frac{\partial E_\mu U}{\partial c(t)})|_{\frac{dc(t)}{dt}=0}$$

This extends Definition 2 to the case where the length of life is not known with certainty, but is described by an exogenous distribution. The rate of discount depends on the mortality pattern considered. Indeed:

Proposition 3 *In the case of the additive utility function, the rate of discount is given by:*

$$RD_\mu^{add}(c,t) = \mu(t) - \frac{\alpha'(t)}{\alpha(t)} \tag{10}$$

For the time neutral utility function, the rate of discount is given by

$$RD_\mu^{tn}(c,t) = \mu(t) - \mu(t)\frac{\int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}{\int_t^{+\infty} d(t_1)\phi'(\int_0^{t_1} u(c(\tau))d\tau)dt_1} \tag{11}$$

Proof. By integrating by parts (9), we find that:

$$E_\mu U(c) = \int_0^{+\infty} s(t)\frac{\partial U(c,T)}{\partial T}|_{T=t}dt$$

where $s(t)$ is the survival function.

In the additive case, $\frac{\partial U^{add}(c,T)}{\partial T}|_{T=t} = \alpha(t)u(c(t))$ and

$$E_\mu U^{add}(c) = \int_0^{+\infty} s(t)\alpha(t)u(c(t))dt \tag{12}$$

which implies that $\frac{\partial E_\mu U^{add}(c)}{\partial c(t)} = s(t)\alpha(t)u'(c(t))$ and $RD_\mu^{add}(c,t) = \frac{-s'(t)}{s(t)} - \frac{\alpha'(t)}{\alpha(t)}$.

In the time neutral case, $\frac{\partial U^{tn}(c,T)}{\partial T}|_{T=t} = u(c(t))\phi'(\int_0^t u(c(t))dt)$, and we find:

$$E_\mu U^{tn}(c) = \int_0^{+\infty} s(t)u(c(t))\phi'\left(\int_0^t u(c(\tau))d\tau\right)dt \tag{13}$$

so that:

$$\frac{\partial E_\mu U^{tn}(c)}{\partial c(t)} = u'(c(t))\left[s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''\left(\int_0^{t_1} u(c(\tau))d\tau\right)dt_1\right]$$

and

$$RD_{\mu}^{tn}(c,t) = \frac{-s'(t)\phi'(\int_0^t u(c(\tau))d\tau)}{s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1} \tag{14}$$

or also:

$$RD_{\mu}^{tn}(c,t) = \mu(t) - \frac{\mu(t)\int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}{s(t)\phi'(\int_0^t u(c(\tau))d\tau) + \int_t^{+\infty} s(t_1)u(c(t_1))\phi''(\int_0^{t_1} u(c(\tau))d\tau)dt_1}$$

which, after integration by parts of the denominator of the fraction, gives (11). ■

In the additive case, the rate of discount is the sum of the mortality rate and the rate of time preference, as is well known. In the time neutral case, even though individuals have no pure time preferences, in the typical case where u is positive, ϕ strictly concave and mortality greater than zero, the rate of discount is greater than the hazard rate of death. Bommier (2006) considers realistic mortality rates and exponential or hyperbolic functions ϕ. This leads to discount functions that are approximately exponential or hyperbolic. In fact, by adjusting the functions ϕ and u, any decreasing discount function can be generated. Indeed, taking matters further, we will see in the following theorem that for any given mortality pattern, any additive preferences with non-negative rates of time preference can be obtained as the limit of time neutral preferences.

Theorem 2 *Assume that individuals face an exogenous mortality pattern and that the hazard rate of death is always positive. For any additive preferences that generate positive rates of discount*[5]*, there exists a sequence of time neutral preferences such that the corresponding expected utility functions (equation (9)) converge (weakly and up to positive affine transformations*[6]*) towards the expected utility function obtained from the additive representation.*

Proof. As in equation (5), denote by α and u a pair of discount and instant utility functions that characterize the additive preferences. The corresponding expected utility function, $E_\mu U^{add}$, defined by (9), can be rewritten as in (12). The positivity of the rates of discount implies that $\mu(t) - \frac{\alpha'}{\alpha}(t) > 0$ for all t.

[5] In the additive case the age-specific rates of discount are given by (10). As mortality rates are assumed to be positive, the rates of discount are positive whenever the rates of time preference (equation (7)) are non-negative.

[6] What is meant by "weak convergence up to positive affine transformations" is formalized below (equations (15) and (16)).

For any $\varepsilon > 0$ define U_ε^{tn} by:

$$U_\varepsilon^{tn}(c,T) = \phi_\mu \left(\int_0^T u_\varepsilon(c(t))dt \right)$$

with

$$u_\varepsilon(c(t)) = 1 + \varepsilon u(c(t)) \text{ and } \phi_\mu(x) = \int_0^x (\alpha(t) - \frac{\alpha'(t)}{\mu(t)})dt$$

Because $\alpha > 0$ and $\mu(t) - \frac{\alpha'}{\alpha}(t) > 0$, the function ϕ_μ has a positive derivative. Also $u'_\varepsilon = \varepsilon u' > 0$. Thus, the utility functions U_ε^{tn} represent time neutral preferences.

From (13), we know that the corresponding expected utility function can be written as:

$$E_\mu U_\varepsilon^{tn}(c) = \int_0^{+\infty} s(t) u_\varepsilon(c(t)) \phi'_\mu \left(\int_0^t u_\varepsilon(c(\tau))d\tau \right) dt$$

I show below that, for any consumption paths c_0, c_1, c, such that $c_1(t) > c_0(t)$ for all t,we have:

$$E_\mu U_\varepsilon^{tn}(c_1) - E_\mu U_\varepsilon^{tn}(c_0) > 0 \tag{15}$$

and

$$\lim_{\varepsilon \to 0} \left(\frac{E_\mu U_\varepsilon^{tn}(c) - E_\mu U_\varepsilon^{tn}(c_0)}{E_\mu U_\varepsilon^{tn}(c_1) - E_\mu U_\varepsilon^{tn}(c_0)} \right) = \frac{E_\mu U^{add}(c) - E_\mu U^{add}(c_0)}{E_\mu U^{add}(c_1) - E_\mu U^{add}(c_0)} \tag{16}$$

This is what is meant by "converges weakly up to positive affine transformations". Clearly, these conditions guarantee that at the limit $\varepsilon \to 0$ the expected utility function $E_\mu U_\varepsilon^{tn}$ will represent the same preferences over consumption profiles as $E_\mu U^{add}$.

Inequality (15) is a direct consequence of the fact that the utility functions U_ε^{tn} are increasing in consumption that occurs before death. Equality (16) is shown thereafter using a Taylor expansion in ε. We have:

$$E_\mu U_\varepsilon^{tn}(c) = \int_0^{+\infty} s(t) u_\varepsilon(c(t)) \phi'_\mu \left(\int_0^t u_\varepsilon(c(\tau))d\tau \right) dt$$

Replacing $u_\varepsilon(c,t)$ by $1 + \varepsilon u(c(t))$ and keeping only the zero and first order terms in ε we find:

$$\begin{aligned} E_\mu U_\varepsilon^{tn}(c) \;=\; & \textstyle\int_0^{+\infty} s(t)\phi_\mu(t)dt \\ & +\varepsilon \textstyle\int_0^{+\infty} s(t)u(c(t))\phi'_\mu(t)dt \\ & +\varepsilon \textstyle\int_0^{+\infty} s(t)\phi''_\mu(t) \left(\int_0^t u(c(\tau))d\tau \right) dt \\ & +o(\varepsilon) \end{aligned} \tag{17}$$

The first term is a constant, independent of c and ε, that I denote by A. Switching the order of integration in the third term, we find that:

$$\int_0^{+\infty} s(t)\phi''_\mu(t)\left(\int_0^t u(c(\tau))d\tau\right)dt = \int_0^{+\infty} u(c(t))\left(\int_t^{+\infty} s(\tau)\phi''_\mu(\tau)d\tau\right)dt \tag{18}$$

Proceeding to an integration by parts and using $\phi'_\mu(t) = \alpha(t) - \frac{\alpha'(t)}{\mu(t)}$, we compute:

$$\begin{array}{rcl} \int_t^{+\infty} s(\tau)\phi''_\mu(\tau)d\tau & = & \left[s(t)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right)\right]_t^{+\infty} - \int_t^{+\infty} s'(\tau)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right)d\tau \\ & = & -s(t)\left(\alpha(t) - \frac{\alpha'(t)}{\mu(t)}\right) - \int_t^{+\infty}\left[s'(\tau)\alpha(\tau) + \alpha'(\tau)s(\tau)\right]d\tau \\ & = & s(t)\frac{\alpha'(t)}{\mu(t)} \end{array} \tag{19}$$

Using (18) and (19), and replacing $\phi'_\mu(t)$ by $\alpha(t) - \frac{\alpha'(t)}{\mu(t)}$ in the second term of (17), we eventually obtain:

$$E_\mu U^{tn}_\varepsilon(c) = A + \varepsilon\int_0^{+\infty} s(t)\alpha(t)u(c(t))dt + o(\varepsilon) \tag{20}$$

The first order term is thus precisely $E_\mu U^{add}(c)$ and (16) directly follows from (20). ■

An implication of Theorem 2 is that, when modeling life-cycle behavior under a given mortality pattern, all the predictions of the additive models with non-negative rates of time preference can be reproduced, up to infinitesimally small differences, by time neutral models. Thus, there is no chance to infer from empirical studies that the additive formulation with non-negative rates of time preference is better than the time neutral one, unless heterogeneity in mortality across agents is considered. This point is particularly important since, to my knowledge, the validity of the additive assumption has never been challenged by studies that consider heterogeneity in mortality. In other words, Theorem 2 tells us that, up to now, there is no piece of empirical evidence that can give more credit to the additive model than to the time neutral one.

Interestingly enough, Theorem 2 is not symmetrical. In fact, from equation (13), we see that in the time neutral case, the expected utility function that represents the preferences over consumption profiles is, in general, not additive. Thus, it cannot be obtained as the limit of a sequence of additive expected utility functions. Although the additive and time neutral models have the same degree of complexity, the time neutral models provide a wider class of preferences with positive rates of discount than the additive models, when a given non-degenerate mortality pattern is considered. That is because preferences over consumption profiles under an exogenous mortality pattern do not depend on the rate of substitution between consumption and the length of life in the additive model[7], while they do depend on it in the time neutral model.

8 Sixth Argument: On the Effect of Mortality Changes

In the previous section, we saw that there may be some similarity between the predictions of the time neutral and the additive models on life-cycle behavior under an exogenous mortality pattern. More precisely, I showed that for any given mortality pattern, with positive hazard rates of death, and any additive preferences, with non-negative rates of time preference, I could define a sequence of time neutral utility functions such that the corresponding expected utility functions converge towards the expected utility function obtained with the additive formulation. I could not, however, find a sequence of time neutral utility functions that satisfy this property for all mortality patterns. In other words, although additive and time neutral preferences may give similar predictions when a given mortality pattern is considered, they will predict, in general, contrasted effects of mortality changes.

In particular, a fundamental difference between the two models is that the rate of discount (Definition 4) will react quite differently to mortality. To stress this point, we can examine the Volterra derivative $\frac{\partial RD_\mu(c,t_1)}{\partial\mu(t_2)}$, which gives the effects of a change in mortality around age t_2 on the rate of discount at age t_1:

Proposition 4 *In the additive case:*

$$\frac{\partial RD_\mu^{add}(c,t_1)}{\partial\mu(t_2)} = \delta(t_2-t_1) \text{ where } \delta \text{ is the Dirac delta function}$$

In the time neutral case:

$$\begin{aligned}\frac{\partial RD_\mu^{tn}(c,t_1)}{\partial\mu(t_2)} \quad = \quad & \frac{1}{\mu(t_1)}RD_\mu^{tn}(c,t_1)\delta(t_2-t_1) \\ & +\mu(t_1)\frac{\phi'(\int_0^{t_1}u(c(\tau))d\tau)\int_{t_2}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^\tau u(c(\tau_1))d\tau_1)d\tau}{\left(\phi'(\int_0^{t_1}u(c(\tau))d\tau)+\int_{t_1}^{+\infty}\frac{s(\tau)}{s(t_1)}u(c(\tau))\phi''(\int_0^\tau u(c(\tau_1))d\tau_1)d\tau\right)^2}1_{(t_2>t_1)}\end{aligned} \tag{21}$$

[7] Preferences over consumption profiles provided by the expected utility function shown in equation (12) do not change if a constant is added to u.

where δ is the Dirac delta function and $1_{(t_2>t_1)}$ a dummy that equals one if $t_2 > t_1$, and zero otherwise.

Proof. The result for the additive case is immediate from equation (10). For the time neutral case, the result is also obvious from equation (11) when $t_2 \leq t_1$. The only difficult case is when $t_2 > t_1$. In this instance, equation (14) can be rewritten as:

$$RD_\mu^{tn}(c,t) = \mu(t_1)\frac{\phi'(\int_0^{t_1} u(c(\tau))d\tau)}{\phi'(\int_0^{t_1} u(c(\tau))d\tau) + \int_{t_1}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{t_1} u(c(\tau_1))d\tau_1)d\tau} \tag{22}$$

Note that $\frac{s(\tau)}{s(t_1)} = \exp(-\int_{t_1}^{\tau} \mu(t)dt)$. So we have:

$$\begin{aligned} \frac{\partial\left(\frac{s(\tau)}{s(t_1)}\right)}{\partial\mu(t_2)} &= -\frac{s(\tau)}{s(t_1)} \text{ for } t_1 < t_2 < \tau \\ \frac{\partial\left(\frac{s(\tau)}{s(t_1)}\right)}{\partial\mu(t_2)} &= 0 \text{ for } t_1 < \tau < t_2 \end{aligned}$$

This implies that for $t_2 > t_1$:

$$\begin{gathered} \frac{\partial}{\partial\mu(t_2)}\left(\int_{t_1}^{+\infty} \frac{s(\tau)}{s(t)} u(c(\tau))\phi''(\int_0^{t_1} u(c(\tau_1))d\tau_1)d\tau\right) = \\ -\int_{t_2}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau \end{gathered}$$

which explains why we obtain (21) by taking the derivative of (22) with respect to $\mu(t_2)$. ■

In the additive case, the result is very simple: an increase in the hazard rate of death at age t_2 of $\delta\mu$ causes an increase in the rate of discount at age t_2 of $\delta\mu$, and has no impact on the rate of discount at other ages. This is because the rate of discount is simply the sum of the hazard rate of death and an exogenous parameter.

In the time neutral case, the result is very different. In fact, there are two fundamental differences. First, an increase in the hazard rate of death at age t_2 affects positively and in *the same proportion* the rate of discount at age t_2 (first term in equation (21)). In other words, the elasticity of the rate of discount at age t_2 with respect to the hazard rate of death at age t_2 equals 1. Second, a change in the hazard rate of death at time t_2 will affect the rate of discount at all ages smaller than t_2. More precisely, if the hazard rate of death increases of $\delta\mu$ between ages t_2 and $t_2 + dt$ then, for all ages $t_1 < t_2$, the rate of discount will change from $RD^{tn}(t_1)$ to:

$$RD^{tn}(t_1) + \frac{\int_{t_2}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau}{\phi'(\int_0^{t_1} u(c(\tau))d\tau) + \int_{t_1}^{+\infty} \frac{s(\tau)}{s(t_1)} u(c(\tau))\phi''(\int_0^{\tau} u(c(\tau_1))d\tau_1)d\tau} RD^{tn}(t_1) \times dt\delta\mu$$

If u is positive and ϕ strictly concave (that is, if individuals are willing to live longer and are temporally risk averse), the adjustment is negative. Thus, in that case, the time neutral model predicts that an increase in the mortality rate at age t_2 will have a positive impact on the rate of discount at age t_2 and a negative impact on the rate of discount at all ages before t_2.

An intuitive interpretation of the results of Proposition 4 can be given. Mortality actually generates two kinds of risk. A risk on consumption (consumption is contingent on survival) and a risk on lifetime utility (lifetime utility is low in the case of an early death and high in the case of a late death). In both the additive and time neutral models, the risk on consumption affects the discount rates in the simplest way: mortality rate at age t contributes additively to the rate of discount at age t (this explains the first terms of equations (10) and (11)). The risk on lifetime utility has no effect in the additive model because of the underlying assumption of temporal risk neutrality. In the time neutral model, when ϕ is strictly concave, individuals exhibit temporal risk aversion. That incites them to re-allocate consumption towards young ages in order to decrease the risk on lifetime utility. Indeed, by consuming early in the life cycle, individuals avoid the low levels of lifetime utility that would result from having a short life with low levels of consumption. In other words, they see the intertemporal allocation of consumption as a way to (partially) insure themselves against the risk of death. But the need for insurance at a given age results from two parameters: (1) the probability of incurring damage (death, in the present case) at that age and (2) the magnitude of the damage (the expected quantity of future pleasures in case of survival: or, in a first approximation, the life expectancy at that age). Mortality affects both parameters, but in opposite directions. It enhances the probability of damage, but diminishes the magnitude of the damage. More precisely, mortality at age t increases the probability of incurring damage at age t and decreases the magnitude of the damage in case of death before age t. The first point explains why the second term of (11) (and hence the rate of discount at age t) increases with mortality at age t. The second point clarifies why an increase in the mortality rate at age t also causes a decrease in the rate of discount at all ages under t.

In practice, we would like to know what happens when there is a global mortality decline that is characterized by a decrease in mortality rates at all ages. According to the additive model, the result is unambiguous: such a global mortality decline implies a decline in the rate of discount at all ages. This is no longer true in the time neutral model. In this latter model, in the typical case where u is positive and ϕ is strictly concave, such a global mortality decline may have a positive or a negative impact on the age-specific rates of discount. Indeed, the rate of discount at an age t was shown to depend positively on the mortality rate at age t and negatively on the mortality rates at ages greater than t. There are, therefore, two opposing effects, which can aggregate into a positive or a negative effect. The computations based on historical mortality rates that will be provided in subsection 8.1 show examples of both positive and negative aggregate effects. Thus, we know that it is impossible to provide a general result on the impact of a global mortality decline on the rates of discount for the time neutral model. Some interesting results can, however, be obtained if additional assumptions are made on how age specific mortality rates are affected by a global mortality decline:

Proposition 5 *Consider two mortality patterns described by hazard rates of death $\mu_1(t)$ and $\mu_2(t)$, with:*

$$\frac{\mu_2(t)}{\mu_1(t)} \leq \frac{\mu_2(t')}{\mu_1(t')} \leq 1 \text{ for } t \leq t' \tag{23}$$

Then, for all consumption paths such that $u(c(t)) > 0$ for all t, we have:

$$RD^{tn}_{\mu_2}(c,t) \leq RD^{tn}_{\mu_1}(c,t) \text{ for all } t.$$

Moreover, if, in addition, ϕ is strictly concave, $\frac{-\phi''}{\phi'}$ non-increasing and mortality non-decreasing with age, for all constant consumption paths such that $u(c) > 0$, we have:

$$RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq RD^{add}_{\mu_1}(c,t) - RD^{add}_{\mu_2}(c,t) = \mu_1(t) - \mu_2(t) \quad \text{for all } t$$

Proof. See appendix B. ■

According to the first point of Proposition 5, if we consider a “high mortality” context (μ_1) and a "low mortality" context (μ_2), such that mortality is higher at all ages in the "high mortality" context and the relative difference in mortality rates, $|\log(\frac{\mu_1}{\mu_2})|$, decreases with age, we know that the time neutral model will predict higher rates of discount in the "high mortality" context.

Moreover, the second point of Proposition 5 indicates that if $\frac{-\phi''}{\phi'}$ is positive and non-increasing[8] and if mortality is increasing with age[9], the difference in the rates of discount will exceed the differences in the mortality rates. That means that the rates of discount are, in that case, more sensitive to mortality in the time neutral model than in the additive model.

Interestingly enough, the results of Proposition 5 can be compared with the findings of empirical studies on heterogeneity in discount rates. Indeed, differential mortality has been quite well documented by demographic studies. It is well known that in the USA, being a woman, or being rich, educated or white are factors that are negatively correlated with mortality[10]. Moreover, it is also often found that whatever the socioeconomic status considered (e.g gender, education, etc.), the differential mortality, measured by the absolute value of the difference in the log of mortality rates, tends to decrease with age after ages 30 or 40. Thus, from Proposition 5, according to the time neutral model, we expect to find that in the USA, women, rich, educated and white individuals have lower values of $RD_\mu - \mu$ (the difference between the rate of discount and the mortality rate). Conversely, the additive model predicts that $RD_\mu - \mu$ should be the same across the population.

Two well-known empirical studies concur with the predictions of the time neutral model. Lawrance (1991), who used data from the PSID, found that the rate of discount is negatively correlated with education, wealth and being white[11]. Moreover the differences in the rates of discount she observed are much larger than the differences in mortality rates[12]. Warner and Pleeter (2001), who analyzed how US military service-

[8] This is equivalent to stating that individuals provided with a constant consumption profile exhibit a positive and non-increasing risk aversion with respect to life duration.

[9] Demographic studies show that this is generally the case after age 25.

[10] See for example the data provided by the Berkeley Mortality Database for comparison by gender or by race, and the results of Brown, Liebman and Pollet (2002) for data on differential mortality by gender, race and education.

[11] Lawrance used household data and did not explore the role of gender.

[12] Remember that in a country such as the USA, the mortality rate is only about 0.2 % at age 40 and does not reach 1% before age 60. Differences in age-specific mortality rates across socio-economic groups are typically a fraction of a percent and much smaller than the differences in the rates of discount found by Lawrance (which are of a few percent).

men chose between lump-sum payments and pensions, found that men, less educated people, blacks and those with low incomes had higher rates of discount. They also found a heterogeneity in the rates of discounts that largely exceeds the differences in mortality rates. These findings are consistent with the time neutral model, while they cannot be explained by the additive model, without introducing further assumptions on the relation between mortality and the discount function[13].

8.1 An Illustration Using Historical Mortality Rates

The recent history of developed countries is characterized by a huge decline in mortality rates. In order to show how important the difference between the additive models and the time neutral models can be when considering historical mortality decline, we conduct the following two exercises.

8.1.1 Example 1

Imagine that in 1937, the year in which Samuelson's paper on the Discounted Utility Model was published (Samuelson, 1937), we observed that individuals of age 30 exhibited a rate of discount of 4% and explore the three following possibilities:

- Case A (Additive preferences): This rate of discount is due to the fact that individuals had additive preferences and expected to die according to the average age-specific mortality rates observed in the USA in 1937.
- Case B (Time neutral preferences with a constant absolute risk aversion with respect to length of life): This rate of discount is due to the fact that individuals had time neutral preferences with a function ϕ of the form $\phi_1(x) = \frac{1-e^{-kx}}{k}$, and that they expected to have a constant quality of life and to die according to the mortality rates of 1937.
- Case C (Time neutral preferences with a constant relative risk aversion with respect to length of life): This rate of discount is due to the fact that individuals had time neutral preferences, with a function ϕ of the form $\phi_2(x) = \frac{x^{1-\kappa}}{1-\kappa}$, and

[13] It could be argued, for example, that the discount function $\alpha(\cdot)$ is related to morbidity, and decreases more rapidly for individuals who have higher mortality rates.

expected to have a constant quality of life and to die according to the mortality rates of 1937.

Now, let us ask the following question: in each case, what would have been these individuals' rates of discount if they had expected to face the mortality rates observed in subsequent years? In solving this problem, we find what the effect of mortality decline on the rate of discount at age 30 would have been if individuals' preferences had remained the same.

In practice, I used the historical cross-sectional mortality rates provided by the Berkeley Mortality Database. As shown in Figure 1, the mortality rate at age 30 decreased rapidly between 1937 and 1960. Between 1960 and 2000, the mortality rate at age 30 had a non-monotonic evolution, but its global trend indicates a slow decline. Life expectancy at age 30 increased during the whole period (Figure 2).

For our exercise, I calibrated the rate of time preference (for case A), the function ϕ_1 (for case B) and the function ϕ_2 (for case C), so that the rate of discount of a 30 year-old individual was of 0.04 per year with the mortality of 1937. Then, for each year from 1938 to 2000, I computed the rate of discount that followed from the mortality observed in those years.

The results are shown in Figure 3. We know from Proposition 4 that in the case of additive preferences, the rate of discount is just the sum of the mortality rate and the rate of time preference. Thus, the solid line that gives the rate of discount in the additive case exactly follows the evolution of the mortality rate shown in Figure 1. However, as the mortality rate is very small compared to the rate of time preference (note that the scales of Figures 1 and 3 differ by a factor of 10), the rate of discount is found to decrease only very slightly. It equals 0.03754 in 1960 and 0.03739 in 2000.

The two dashed lines, which represent the time neutral preferences, show radically different patterns. In Case B, the mortality decline that occurred between 1937 and 1960 leads to a drop of 0.01743 in the rate of discount. That is 7.1 times greater than what we would have predicted using the additive model! This is due to the major decline in the mortality rate at age 30. After 1960, the rate of discount goes up and down, but the average trend shows a slight increase. Thus, during this period, the evolution of the rate of discount shows a global trend that does not follow the evolution of the mortality rate. In fact, during the period from 1960-2000, the mortality rate at age 30 declined only slightly while life expectancy considerably. I explained after Proposition 4 that in the time neutral model, the rate of discount at age 30 is linked to mortality through two different channels. It is positively related to the mortality rate at age 30, and negatively related to the mortality rate at older ages. We see from our results that during the period from 1937 to 1960, it is the first factor that dominates, while after 1960, if we look at the global trend, it is the second one that predominates.

The results in Case C are comparable to those in Case B, although they further diverge from the results of the additive model. The interpretation is similar to Case B.

Overall, we found that the time neutral model can lead to radically different predictions of the impact of mortality decline. A drop of 1.743 % or of 1.928 % in the rate of discount at age 30 between 1937 and 1960, as we respectively found in Cases B and C, is likely to generate a large impact on savings, human capital investment, and henceforth, on economic growth. The additive model would have predicted a drop in the rate of discount of only 0.25 %.

8.1.2 Example 2

To deal with more concrete issues, let us look at consumption smoothing behaviors. Consider the case of an individual who earns 20000 dollars a year between ages 20 and 60 and nothing afterwards. Assume that there are perfect annuity markets and only one riskless asset whose rate of return equals 3.5% per year. How would such an individual smooth consumption and save along the life cycle? Let us consider three specifications for individuals' preferences:

$$\begin{aligned}
&\text{1 - Additive model} &:\;& U^{add} = \int_0^T e^{-\rho t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt \\
&\text{2 - Time neutral model (CARA)} &:\;& U^{tn}_{cara} = 1-\exp\left(-k\int_0^T \frac{c(t)^{1-\gamma}-c_{cara}}{1-\gamma}dt\right) \\
&\text{3 - Time neutral model (CRRA)} &:\;& U^{tn}_{crra} = \frac{1}{1-\kappa}\left(\int_0^T \frac{c(t)^{1-\gamma}-c_{crra}}{1-\gamma}dt\right)^{1-\kappa}
\end{aligned}$$

For each specification, we can compute the optimal life cycle behavior for two different mortality patterns[14]. The first one is given by the mortality rates that were observed

[14] In all three specifications, the intertemporal elasticity of substitution, $\frac{1}{\gamma}$, was set at 0.9. The constants ρ ,k, κ, c_{cara} and c_{crra} were chosen so that, with 1950 mortality, a 40 year old individuals have a rate of discount of 0.03 per year and a Value of a Statistical Life of 4 million dollars. Consumption before age 20 is assumed to be exogenous and equal to 16000 dollars per year. The optimal consumption

in 1950 in the USA. The second one corresponds to the 2000 mortality rates. The predicted age-specific consumption and wealth profiles are shown in Figure 4.

When preferences are additive, the optimal consumption profile has the same shape, whether we consider 1950 or 2000 mortality rates. The "2000 consumption" is obtained from the 1950 one by a simple scaling down. It is in fact well-known that, with perfect annuity markets and a constant intertemporal elasticity of substitution, the rate of consumption growth is independent of mortality rates. Consumption is lower with 2000 mortality rates, because longevity extension generates a dilution effect.

The time neutral specifications suggest very different pictures. Firstly, the 2000 consumption and 1950 consumption no longer have the same shape. In both the CARA and the CRRA cases, 2000 consumption lies below 1950 consumption at young ages, and above at old ages. This reflects the fact that mortality decline has a two-fold effect. Firstly, there is a dilution effect, as in the additive case. Secondly, and here is the novelty, there is a significant impatience effect.

To see how significant is the divergence in predictions, we can consider individuals' wealth at retirement. The additive specification suggests that wealth at retirement increases by 14% when passing from 1950 to 2000 mortality rates. Rational individuals increase their savings because the retirement period becomes longer. However, the time neutral specifications suggest much larger increases (26% for the CARA case and 28% for the CRRA case). Even in such a rough example, where retirement age does not adapt to mortality decline, accounting for the change in impatience appears to be as important as accounting for the extension of the retirement period.

9 The time neutral model in practice

One attractive feature of Yaari's model is its mathematical tractability. With Yaari's model, preferences over consumption profiles, conditional on an exogenous mortality pattern, are represented by the utility function:

$$E_{\mu}^{add}(c) = \int_{0}^{+\infty} s(t)\alpha(t)u(c(t))dt$$

profiles were numerically computed with the method detailed in Section 9.

Dealing with such a utility function is, technically speaking, particularly convenient for several reasons. First, preferences over consumption after age t are independent of consumption before age t. Therefore, in a dynamic setting, individuals need not remember the past to have time consistent behaviors. Moreover the additive structure of the expected utility function often leads to relatively simple optimization problems. A number of life cycle problems (e.g. consumption smoothing, portfolio choices) can be studied with standard techniques, such as dynamic programming, and, for particular functions u, yield to simple solutions.

The object of this section is to discuss how the landscape is transformed when working with the time neutral model. It will be split into three parts. A first subsection points at the technical difficulties that emerge when dealing with the time neutral model. As we will see, there are no fundamental obstacles for using standard techniques, such as dynamic programming. The main difference, however, is that explicit solutions cannot readily be found. Nevertheless, it is possible to work with the time neutral model without developing cumbersome numerical computations. First, as will be explained in Subsection 9.2, a linear approximation makes it possible to retrieve all the simplicity of the additive model, while maintaining key aspects of the time neutral model. Secondly, Subsection 9.3 provides a very simple method for numerically computing exact solutions when there are complete markets.

9.1 History dependence and dynamic programming

Agents with time neutral preferences who face an exogenous mortality pattern have preferences over (stochastic) consumption profiles represented by the utility function:

$$E_\mu U^{tn}(c) = \int_0^{+\infty} s(\tau)u(c(\tau))\phi'\left(\int_0^{\tau} u(c(\tau'))d\tau'\right) d\tau \tag{24}$$

A noteworthy difference with the additive formulation is that preferences over consumption after time t generally depend on consumption prior to t. From (24), given a consumption profile $\widetilde{c}$ between times 0 and t, preferences over consumption profiles after time t are represented by the utility function:

$$\int_t^{+\infty} s(\tau)u(c(\tau))\phi'\left(H_t + \int_t^{\tau} u(c(\tau'))d\tau'\right) d\tau \tag{25}$$

where

$$H_t = \int_0^t u(\widetilde{c}(\tau))d\tau$$

is the "stock of felicity" that has been accumulated up to time t. In a dynamic setting, under the assumption of time consistency, the utility function (25) represents the preferences of an agent of age t with past consumption $\widetilde{c}$. Preferences may then exhibit history dependence, since past consumption affects H_t which enters into the agents' utility functions.

At this point, however, it is useful to distinguish the case where ϕ' is exponential from the general case. When $\phi'(x) = e^{-kx}$, past consumption only matters in (25) through a positive multiplicative factor, e^{-kH_t}, and therefore has no impact on individual preferences. Thus, precisely as with the additive model, individuals do no need not to remember the past to be time consistent. In fact, when ϕ' is exponential, preferences represented by (6) are stationary (see Bommier, 2005) and the utility function (24) is a particular case of stochastic differential utilities that are considered by Duffie and Epstein (1992).

When ϕ' is not exponential preferences over consumption after age t depend on consumption before t. Thus, in order to have time consistent behaviors, individuals have to bear in mind some of their history[15]. History dependence however takes a very simple form. Preferences over consumption after age t depend on the past only through H_t, the stock of felicity that has been accumulated at age t. This largely resembles habit formation problems, where preferences at age t depend on the past only through the stock of habits that has been accumulated at time t. Dynamic programming can then be implemented in a standard way, even though the technical problems that one has to face are indisputably more complex. The dynamics involves two scalar state variables (wealth and the stock of past felicity) instead of one (wealth) with the usual additive case. For numerical applications going from one to two state variables only represents a slight increase in complexity. Gomes and Michaelides (2003) and Polkovnichenko (2005) produced papers on portfolio choice with habit formation that successfully deal with similar (and significantly greater) technical difficulties. Their approach could be replicated. Alternatively, one may opt for the simpler approaches developed below.

[15] Another route, pursued in Bommier (2006), involves assuming the history independence of preferences and allowing for time-inconsistencies.

9.2 A linear approximation of the time neutral

From Theorem 2, we know that all the predictions of Yaari's model can be reproduced, up to infinitesimally small differences. Thus it must be the case that for some specification the time neutral model has the same tractability as the additive specification. The strategy, suggested in Bommier (2006), involves assuming that consumption remains in a range $[c_{\min}, c_{\max}]$, such that the difference in welfare between having a high or a low level of consumption is much smaller than the difference of welfare between being alive with a low level of consumption and being dead[16]:

$$\frac{u(c_{\max}) - u(c_{\min})}{u(c_{\min}) - 0} << 1$$

For any c^* in $[c_{\min}, c_{\max}]$ one can write

$$u(c) = u(c^*)[1 + \varepsilon v(c)]$$

with $\varepsilon = \frac{u(c^*)-u(c_{\min})}{u(c^*)} << 1$ and $v(c) = \frac{u(c)-u(c^*)}{u(c^*)}$. The idea is then to approximate the utility function (24) by a first order approximation in ε. Following the lines of the proof of Theorem 2, one can compute:

$$E_\mu U^{tn} \simeq A + \varepsilon \int_0^{+\infty} s(t)\alpha_\mu(t)v(c(t))dt$$

where A is a constant and α_μ is a discount function given by

$$\alpha_\mu(t) = \frac{1}{s(t)} \int_t^{+\infty} d(\tau)\phi'(\tau u(c^*))d\tau$$

Thus, individuals approximately behave as if they were maximizing the expectation of:

$$\int_0^{+\infty} s(t)\alpha_\mu(t)v(c(t))dt$$

We are then back to an additive specification and retrieve all the tractability of Yaari's formulation. The fundamental difference with Yaaris's formulation is that the discount function is now related to mortality. This is of course of crucial importance for studying

[16] This actually involves assuming that the value of life is very large. In fact, the linear approximation developped below corresponds to the limit case where the value of life is infinite.

the role of mortality changes.

Such an approximation preserves one of the main features of the time neutral model (the strong relation between mortality and impatience) and proves to be pretty efficient for studying the impact of mortality on consumption smoothing[17]. However, by "forcing additivity", we necessarily loose some features of the time neutral model, as its ability to separate risk aversion and intertemporal elasticity of substitution. This linear approximation will then be less advisable to study life cycle portfolio choices, since it would lead to the same shortcomings as the standard additive case[18].

9.3 Numerical solutions

In this section we explain how optimal life cycle behavior can be very readily and quickly numerically computed when financial markets are complete. We give accounts of the method without addressing the technical questions as to the conditions that would ensure this method's efficiency.

Following the martingale approach (see Duffie, 2001, for example), when financial markets are complete, life-cycle optimization is equivalent to finding the consumption process c that solves:

$$\max_c E\left[E_\mu U^{tn}(c)\right] \quad \text{subject to } W = E\left[\int_0^{+\infty} p(t)c(t)dt\right] \tag{26}$$

where p is a contingent price process. Rather than using (24), it proves more convenient to use the equivalent representation:

$$E_\mu U^{tn}(c) = \int_0^{+\infty} d(T)\phi\left(\int_0^T u(c(\tau))d\tau\right) dT$$

It is clear that, when the function ϕ and u are concave, $E_\mu U^{tn}$ is concave. Resolution of the maximization problem (26) can therefore be achieved using standard numerical methods of convex optimization, as described in Boyd and Vandenberghe (2004). How-

[17] For example, if we use this additve approximation to study the example developed in Section 8.1.2, we find that switching from 1950 to 2000 mortality should induce an increase of wealth at retirement by 28%, in the CARA case, and 30% in the CRRA case. These predictions are close to those obtained from an exact resolution of the time neutral (26% and 28%) and sharply contrast with those of the additive model (14%).

[18] As explained in Bommier and Rochet (2006), when preferences are not additively separable, as in the time neutral model, the optimal degree of risk taking varies along the life cycle. This is an intersting feature that would be lost with the linear approximation.

ever, given the particular structure of the objective function it generally proves easier to solve the optimization problem using the utility gradient approach[19]. Basically, one has to compute the gradient of the utility function and to invert it. In the present case, the utility gradient admits a simple expression. Under regularity conditions on the functions ϕ and u, for any "small perturbation process" δc:

$$E\left[E_\mu U^{tn}(c+\delta c) - E_\mu U^{tn}(c)\right] \simeq E\left[\int_0^{+\infty} d(T)\left(\int_0^T u'(c(t))\delta c(t)dt\right)\phi'\left(\int_0^T u(c(\tau))d\tau\right)dT\right]$$

Switching the order of integration:

$$E\left[E_\mu U^{tn}(c+\delta c) - E_\mu U^{tn}(c)\right] \simeq E\left[\int_0^{+\infty} \delta c(t)\pi(t)d\tau\right]$$

with:

$$\pi(t) = u'(c(t))\int_t^{+\infty} d(T)\phi'(\int_0^T u(c(\tau))d\tau)dT$$

The first order conditions of the optimization problem (26) are thus:

$$u'(c(t))\int_t^{+\infty} d(T)\phi'(\int_0^T u(c(\tau))d\tau)dT = \lambda p(t) \text{ for all } t \tag{27}$$

The core of the problem consists in inverting this equation, that is to obtain $c(.)$ from $\lambda p(.)$. Denoting $z(t) = \log(u'(c(t)))$ the problem is to find a fixed point of the mapping:

$$\Omega : \begin{cases} z \in C(\mathbb{R}^+, \mathbb{R}) \to \Omega[z] \in C(\mathbb{R}^+, \mathbb{R}) \\ \Omega[z](t) = \log(\lambda p(t)) - \log\left(\int_t^{+\infty} d(T)\phi'\left(\int_0^T g(z(\tau))d\tau\right)dT\right) \end{cases}$$

where $g = u \circ [u']^{-1} \circ \exp$. Here $[u']^{-1}$ denotes the reciprocal of u' and $\circ$ the composition operator[20].

Now, remark that:

Lemma 1 *Assume that there is a maximal length of life ω (so that $d(T) = 0$ for all $T > \omega$) and that the functions u and ϕ are increasing and concave. Then, for any z_1, z_0*

[19] A detailed account of this approach can be found in Duffie (2001).

[20] In the standard isoelastic case, $u(c) = \frac{c^{1-\gamma} - c_0^{1-\gamma}}{1-\gamma}$, we have $g(z) = \frac{e^{-\frac{1-\gamma}{\gamma}z} - c_0^{1-\gamma}}{1-\gamma}$.

such that $[u']^{-1}\left(e^{z_1(t)}\right)$ *and* $[u']^{-1}\left(e^{z_0(t)}\right)$ *are in* $[c_m,c_M]$ *for all* $t \in [0,\omega]$ *we have:*

$$\sup_{t\in[0,\omega]} ||\Omega[z_1](t) - \Omega[z_0](t)|| \leq K \sup_{t\in[0,\omega]} ||z_1(t) - z_0(t)||$$

with

$$K = \left[\sup_{c\in[c_m,c_M]} \frac{-u'(c)}{cu''(c)}\right]\left[\sup_{\kappa\in[\omega u(c_m),\omega u(c_M)]} \frac{-\kappa\phi''(\kappa)}{\phi'(\kappa)}\right]\left[\sup_{c\in[c_m,c_M]} \frac{cu'(c)}{u(c)}\right] \tag{28}$$

Proof. See appendix C ■

In the three factors that enter into (28), we recognize the intertemporal elasticity of substitution ($\frac{-u'(c)}{cu''(c)}$), the relative risk aversion with respect to length of life $\left(\frac{-\kappa\phi''(\kappa)}{\phi'(\kappa)}\right)$, and the elasticity of instantaneous utility function $\left(\frac{cu'(c)}{u(c)}\right)$. This latter elasticity is the main determinant of the value of life. The smaller $\frac{cu'(c)}{u(c)}$ the greater the marginal rate of substitution between length of life and consumption (and the greater the value of life).

Empirical works suggest that the intertemporal elasticity of substitution is not far from 1. Thus, when relative risk aversion with respect to length of life is small enough, or when the value of life large enough, we have $K < 1$. The mapping Ω is then a contraction and its fixed point can be found by a simple iteration process.

The strategy to solve the optimization problem (26) is then as follows. Step 1: for any λ, find the consumtion process c_λ that solves the first order conditions (27) by computing the fixed point of Ω by iteration. Step 2: compute $W_\lambda = E\left[\int_0^{+\infty} p(t)c_\lambda(t)dt\right]$ and look for the value of λ such that $W_\lambda - W$ equals zero[21].

Moreover, when u is isoelastic (that is when $u(c) = \frac{c^{1-\gamma}-c_0^{1-\gamma}}{1-\gamma}$), the function $[u']^{-1}$ is homogenous, which makes it possible to merge steps 1 and 2 into a single fixed point search. Resolution of (26) involves finding the fixed point that solves $z = \widehat{\Omega}[z]$ where:

$$\widehat{\Omega}[z](t) = \gamma\log\left(E\left[\int_0^{+\infty} p(t)\exp(-\frac{1}{\gamma}\widetilde{\Omega}[z](t))\right]\right) - \gamma\log(W) + \widetilde{\Omega}[z](t)$$

[21] Remark that c_λ solves (26) when W is replaced by W_λ. That means that λ is the marginal utility of wealth when wealth equals W_λ. Thus λ and W_λ are negatively related when $E_\mu U^{tn}$ is concave (which is the case when ϕ and u are concave). Solving $W_\lambda - W = 0$ involves then finding the zero of a decreasing function.

with:

$$\widetilde{\Omega}\left[z\right](t) = \log(p(t)) - \log\left(\int_t^{+\infty} d(T)\phi'\left(\int_0^T \frac{e^{-\frac{1-\gamma}{\gamma}z(\tau)} - c_0^{1-\gamma}}{1-\gamma}d\tau\right)dT\right)$$

In practice, nothing guarantees that the constant K, given by (28), is smaller than one. The suggested iterative method does not necessarily work for all parameters values. Still, with the parameters that allows standard estimates of the rate of discount and the value of a statistical life to be matched, it proved to be extremely efficient[22]. The consumption profiles shown in Figure 4 were computed in such a way in less than a second.

10 Discussion

In his pioneering book that gave birth to life cycle theory, Fisher wrote:

> "shortness of life tends powerfully to increase the degree of impatience, or rate of time preference, beyond what it would otherwise be" (Fisher 1930, p. 85).

Fisher's intuition could not be formalized in the earliest developments of life-cycle theory since they did not account for uncertainty (and therefore for mortality risks). Mortality risks were first considered by Yaari in a paper that has become the model of reference in the literature on intertemporal choice under uncertain lifetime. Yaari, however, made two very particular choices: one that concerns ordinal properties of the utility function, and another relating to its cardinal properties.

On the ordinal side, Yaari chose to consider an additive separable utility function with time preferences. He called it the *Fisher utility function* to stress the relation between his formal work and Fisher's ideas. But Yaari took an important step. In Fisher's mind, time preference and mortality were indissociable. In Yaari's work, time preference became an exogenous element, unrelated to mortality.

[22] Moreover when the application Ω fails to be a contraction mapping, one may look at the application

$$z \to \Omega_\theta[z] = \theta z + (1-\theta)\Omega[z]$$

for some $0 < \theta < 1$. The application Ω_θ has indeed the same fixed point as Ω. However, since Ω is decreasing, Ω_θ is likely to be a contraction mapping even if Ω is not a contraction.

Mortality being a risk, its impact has to governed by cardinal assumptions. However, for cardinal aspects, Yaari simply assumed that the von Neumann-Morgenstern utility function was the same as the Fisher ordinal utility function. Many other choices would have been possible. Any increasing transformation of the Fisher utility function would represent the same ordinal preferences, but have different cardinal properties. Why then take the Fisher utility function and not the logarithm, the cube, the exponential of it? Yaari's choice, which he did not discuss, in fact consisted in making an assumption of temporal risk neutrality. A major consequence of this choice is that the rate of time discounting equals the sum of the rate of time preference and the mortality rate. Since mortality rates are typically much lower than observed rates of discount, Yaari's model eventually provides a theory where time discounting owes very little to mortality. In particular, it rules out Fisher's intuition about the strong relation between impatience and mortality. In reality, Fisher's intuition was banned from mainstream economic theory. During the 40 years that have passed since the publication of Yaari's paper, the economic literature on time preference has grown extraordinarily, ramified[23], but very little has been said on the role of mortality[24].

The present paper shows that a different path, which is no more complex than the one followed by Yaari, might have been pursued. It consists in assuming that individuals have no time preferences but exhibit temporal risk aversion, giving the "time neutral model". With this model, impatience has no ordinal origins, but results from the combination of mortality risks and the cardinal notion of temporal risk aversion. Can such a model be a reasonable option?

Firstly, I discussed the theoretical properties of the time neutral model. In many respects these are similar to those of Yaari's model. Both models were actually derived from a common axiomatic formulation. They are equally complex. The only difference is that one allows for time preference while the other allows for temporal risk aversion.

Secondly, I formally showed that the time neutral model can reproduce all the predictions of Yaari's model, as long as heterogeneity in mortality across agents is ignored. However, to my knowledge, Yaari's model has never been challenged by studies that

[23] See the critical review by Frederick, Loewenstein and O'Donoghue (2002).

[24] A notorious exception is that of Becker and Mulligan (1997) that argues that the rate of time preference decreases with horizon length. There is however no uncertainty (and therefore no mortality risks) in Becker and Mulligan's paper.

used heterogeneity in mortality. Thus, today, there is no empirical evidence indicating that Yaari's model is better than the time neutral model. In particular, the fact that Yaari's formulation proved useful to study consumption patterns, saving behaviors, labor supply, etc. under an uncertain lifetime cannot be considered as an argument supporting Yaari's model. The time neutral model would do at least as well. It may even do a better job since it offers flexibilities that Yaari's model excludes (in particular the time neutral model allows intertemporal elasticity of substitution and relative risk aversion to be disentangled).

The choice between one or the other formulation is however crucial when variations in mortality are considered. In Yaari's model, impatience, measured by the rate of time discounting, is almost exogenous (mortality having a minor role). In the time neutral model, impatience is exclusively driven by mortality. Thus, unsurprisingly, we find that both models have very different predictions about the impact of heterogeneity in mortality. Getting the right intuitions about the role of mortality in the time neutral model requires a rigorous look at the formal expression of the rate of time discounting (Propositions 4 and 5). Even if impatience is driven by mortality, it is not always the case that greater mortality implies greater impatience. In fact, in the time neutral model, mortality in the short term increases impatience while mortality in the long term decreases it. Sufficient conditions were provided for the first effect to dominate the second one, but one should bear in mind that these conditions are not always fulfilled. As a consequence, in order to discriminate between Yaari's model and the time neutral model, it is necessary to have, on the one hand, a very good knowledge of differential mortality (so that heterogeneity in short term mortality and heterogeneity in long term mortality can be compared) and, on the other hand, excellent data on intertemporal choice that make it possible to measure individuals' rates of discount.

The ideal data set does not yet exist. A number of surveys report data on health, health shocks, etc. but it is generally impossible to accurately translate this information into short-term and long-term mortality rates. To my mind, the best available option is to confront the well documented heterogeneity in mortality rates across gender, ethnic, education and income groups with the heterogeneity in discount rates. I explained that this confrontation actually supports the time neutral model over Yaari's model. This certainly does not provide sufficient evidence to abandon the notion of time preference.

However, there are even less arguments in favor of ignoring temporal risk aversion, as is currently done. Accounting for temporal risk aversion is in fact crucial to understanding the transformation that societies are going through along with the rapid evolution of mortality rates.

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APPENDIX

A Proof of Theorem 1

A.1 Necessary Conditions

Let us first prove that if axioms 1-6 are fulfilled, then individuals' preferences can be represented by a utility function that has the properties stated in Theorem 1.

Let us first consider the preferences over consumption profiles, conditional on a given length of life. Note $U(c|T)$ a von Neumann-Morgenstern utility function that represents such preferences. From Axiom 2, we know that preferences over consumption profiles conditional on a length of life T only depend on consumption in $[0, T]$. Moreover, from Axiom 4, we know that preferences are separable. Thus, using Gorman (1968), we know that $U(c|T)$ is of the form:

$$U(c|T) = \psi_T \left(\int_0^T v_T(c(t), t) dt \right) \tag{29}$$

I have indexed the functions ψ_T and v_T using T since there is no reason a priori for these functions to be independent of T. However, let us first remark that through normalization, we can assume without loss of generality that $v_T(1, t) = 1$ for all t and $\frac{\partial}{\partial c} v_T(1, 0) = 1$. Furthermore, from Axiom 4, we know that $\frac{\frac{\partial U(c|T)}{\partial c(t)}}{\frac{\partial U(c|T)}{\partial c(0)}}$ must be independent of T. Using equation (29) and choosing $c(0) = 1$, we find that $\frac{\frac{\partial}{\partial c} v_T(c,t)}{\frac{\partial}{\partial c} v_T(1,0)}$ must, therefore, be independent of T. However, as $\frac{\partial}{\partial c} v_T(1, 0) = 1$ and $v_T(1, t) = 1$ this implies that $v_T(c, t)$ is independent of T. Thus, indexing the function v by T is not necessary.

Now, let us examine the property of the function $U(c, T)$ that can be used to compare lotteries involving lives of different lengths. We know that $U(c, T)$ must give the same preferences as $U(c|T)$ when considering lotteries for which the length of life equals T with certainty. But two von Neumann-Morgenstern utility functions represent the same preferences, if and only if we can obtain one from the other by a positive affine transformation. Thus, we know that there exist two functions $h_1(T)$ and $h_2(T) > 0$ such that:

$$U(c, T) = h_1(T) + h_2(T)U(c|T)$$

Using (29), and defining $\widetilde{\psi}(x, T) = h_1(T) + h_2(T)\psi_T(x)$ we see that $U(c, T)$ is of the form:

$$U(c,T) = \widetilde{\psi}\left(\int_0^T v(c(t),t)dt, T\right) \tag{30}$$

Now compute:

$$\frac{\partial U(c,T)}{\partial T} = v(c(t),t)\widetilde{\psi}'_x(\int_0^T v(c(t),t)dt, T) + \widetilde{\psi}'_T(\int_0^T v(c(t),t)dt, T)$$

and for any $t \leq T$:

$$\frac{\partial U(c,T)}{\partial c(t)} = v'_c(c(t),t)\widetilde{\psi}'_x(\int_0^T v(c(t),t)dt, T)$$

so that:

$$\frac{\frac{\partial U(c,T)}{\partial T}}{\frac{\partial U(c,T)}{\partial c(T)}} = \frac{v(c(T),T)}{v'_c(c(T),T)} + \frac{1}{v'_c(c(T),T)}\frac{\widetilde{\psi}'_T(\int_0^T v(c(t),t)dt, T)}{\widetilde{\psi}'_x(\int_0^T v(c(t),t)dt, T)} \tag{31}$$

Axiom 6 tells us that $\frac{\frac{\partial U(c,T)}{\partial T}}{\frac{\partial U(c,T)}{\partial c(T)}}$ must be independent of consumption at all ages in $[0,T[$. From (31) and Axiom 3, it is possible only if $\frac{\widetilde{\psi}'_T}{\widetilde{\psi}'_x} = k(T)$ for some function k. Define

$$\phi(x,T) = \widetilde{\psi}(x - \int_0^T k(t)dt, T) \tag{32}$$

We have $\phi'_T = -k(T)\widetilde{\psi}'_x + \widetilde{\psi}'_T = 0$. Thus $\phi(x,T)$ is constant in T. With an obvious abuse of notation we write $\phi(x,T) = \phi(x)$ where ϕ is now a single variable function. From (32), $\widetilde{\psi}(x,T) = \phi(x + \int_0^T k(t)dt)$. Defining $u(c,t) = v(c,t) + k(t)$, we obtain from (30):

$$U(c,T) = \phi\left(\int_0^T u(c(t),t)dt\right)$$

Note also that according to Axiom 6, $\frac{\partial}{\partial T}\frac{\frac{\partial U(c,T)}{\partial T}}{\frac{\partial U(c,T)}{\partial c(T)}}|_{\frac{dc(T)}{dT}=0} = 0$. This leads to:

$$\frac{\partial}{\partial T}\left(\frac{u(c,T)}{u'_c(c,T)}\right) = 0$$

which implies that $u(c,T)$ is of the form:

$$u(c,T) = \alpha(T)u(c)$$

for some function α. We are, therefore, left with the general expression for $U(c,T)$:

$$U(c,T) = \phi\left(\int_0^T \alpha(t)u(c(t))dt\right) \tag{33}$$

With this above expression:

$$\frac{\partial U(c,T)}{\partial c(t)} = \alpha(t)u'(c(t))\phi'\left(\int_0^T \alpha(\tau)u(c(\tau))d\tau\right) \tag{34}$$

From Axiom 3 it must therefore be the case that $\alpha u'\phi' > 0$. But, remark that for any $\varepsilon_1, \varepsilon_2 = \pm 1$,

$$\phi\left(\int_0^T \alpha(t)u(c(t))dt\right) = \widetilde{\phi}\left(\int_0^T \widetilde{\alpha}(t)\widetilde{u}(c(t))dt\right)$$

with $\widetilde{\alpha} = \varepsilon_1\alpha$, $\widetilde{u} = \varepsilon_1\varepsilon_2 u$ and $\widetilde{\phi}(x) = \phi(\varepsilon_2 x)$. Thus, by playing with the signs of ε_1 and ε_2, we can always find a utility function of the form (33) with $\alpha > 0$, $u' > 0$ and $\phi' > 0$.

Finally, let us use Axiom 5. From (34) we derive:

$$\frac{\partial U(c,T)}{\partial c^2(t)} = \alpha(t)\phi'\left(\int_0^T \alpha(\tau)u(c(\tau))d\tau\right)u''(c(t))\delta_t + (\alpha(t)u'(c(t)))^2\phi''\left(\int_0^T \alpha(\tau)u(c(\tau))d\tau\right)$$

where δ_t is a Dirac delta function. Thus:

$$\frac{\frac{\partial U(c,T)}{\partial c^2(t)}}{\frac{\partial U(c,T)}{\partial c(t)}} = \frac{u''(c(t))}{u'(c(t))}\delta_t + \alpha(t)u'(c(t))\frac{\phi''}{\phi'} \tag{35}$$

According to Axiom 5, $\frac{\frac{\partial U}{\partial c^2(t)}}{\frac{\partial U}{\partial c(t)}}$ must depend on t only through $c(t)$. However, this is clearly the case for the first term on the right-hand side of (35). Thus, since $u'(c(t)) \neq 0$ (from Axiom 3), we find that $\alpha(t)\frac{\phi''}{\phi'}$ should be independent of t, for all c and T. This is the case only if α is a constant or $\phi'' = 0$.

A.2 Sufficient Conditions

It remains to be shown that if preferences are represented by a von Neumann-Morgenstern utility function (4) with twice continuously differentiable functions α and ϕ such that α, u' and ϕ' are positive and such that $\phi'' = 0$ or/and $\alpha' = 0$, then Axioms 1 to 6 are complied with.

Axioms 1 and 2 are obviously satisfied. Axiom 3 follows from (34). This latter equation also implies that for any c, T and any $t_1, t_2 < T$

$$\frac{\frac{\partial U(c,T)}{\partial c(t_1)}}{\frac{\partial U(c,T)}{\partial c(t_2)}} = \frac{\alpha(t_1)u'(c(t_1))}{\alpha(t_2)u'(c(t_2))}$$

Axiom 4 is thus fulfilled. Also:

$$\frac{\frac{\partial U(c,T)}{\partial T}}{\frac{\partial U(c,T)}{\partial c(T)}} = \frac{u(c(T))}{u'(c(T))}$$

which implies Axiom 6. Last, Axiom 5 follows from (35).

B Proof of Proposition 5

For the first point, rewrite (14) for $i = 1, 2$:

$$RD^{tn}_{\mu_i}(c,t) = \mu_i(t)\frac{\phi'(\int_0^t u(c(\tau))d\tau)}{\int_t^{+\infty} \mu_i(\tau)\exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)\phi'(\int_0^\tau u(c(\tau_1))d\tau_1)d\tau}$$

and use that for all $\tau \geq t$ inequality (23) implies that $\exp\left(-\int_t^\tau \mu_2(\tau_1)d\tau_1\right) \geq \exp\left(-\int_t^\tau \mu_1(\tau_1)d\tau_1\right)$ and $\mu_2(\tau) \geq \mu_1(\tau)\frac{\mu_2(t)}{\mu_1(t)}$ to obtain that $RD^{tn}_{\mu_2}(c,t) \leq RD^{tn}_{\mu_1}(c,t)$.

For the second point, use (11) to write that for $i = 1, 2$

$$RD^{tn}_{\mu_i}(c,t) = \mu_i(t) - \mu_i(t)\frac{\int_t^{+\infty}\exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)u(c(\tau))\phi^{''}(I_\tau)d\tau}{\int_t^{+\infty}\mu_i(\tau)\exp\left(-\int_t^\tau \mu_i(\tau_1)d\tau_1\right)\phi^{'}(I_\tau)d\tau}$$

where $I_\tau = \int_0^\tau u(c(\tau_1))d\tau_1$. Using that $\mu_2(\tau) \geq \mu_1(\tau)\frac{\mu_2(t)}{\mu_1(t)}$ (from inequality (23)) and that $\phi^{''} < 0$ we obtain

$$RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq \mu_1(t) - \mu_2(t) + \mu_1(t)\Delta$$

with

$$\Delta = \frac{\int_t^{+\infty} k(\tau)g(\tau)d\tau}{\int_t^{+\infty} g(\tau)d\tau} - \frac{\int_t^{+\infty} k(\tau)h(\tau)g(\tau)d\tau}{\int_t^{+\infty} h(\tau)g(\tau)d\tau}$$

where $k(\tau) = -\frac{\phi''(I_\tau)}{\phi'(I_\tau)}\frac{u(c)}{\mu_1(\tau)}$, $h(\tau) = \exp\left(-\int_t^\tau(\mu_2(\tau_1) - \mu_1(\tau_1))d\tau_1\right)$ and $g(\tau) = \mu_1(\tau)\exp\left(-\int_t^\tau \mu_1(\tau_1)d\tau_1\right)u(c(\tau))\phi^{'}(I_\tau)$. The functions k, g and h are non-negative. Note also that, by assumption, h is non-decreasing while k is non-increasing.

Thus, Δ is non-negative[25] and $RD^{tn}_{\mu_1}(c,t) - RD^{tn}_{\mu_2}(c,t) \geq \mu_1(t) - \mu_2(t)$. The fact that $RD^{add}_{\mu_1}(c,t) - RD^{add}_{\mu_2}(c,t) = \mu_1(t) - \mu_2(t)$ is a direct consequence of (10). The proof of Proposition 5 is then complete.

C Proof of Lemma 1

For any given $\lambda \in [0,1]$, and any t define $\theta(\lambda) = \Omega[z_0 + \lambda(z_1 - z_0)](t)$. We have $\Omega[z_1](t) - \Omega[z_0](t) = \theta(1) - \theta(0)$ and from the mean value theorem $\Omega[z_1](t) - \Omega[z_0](t) = \theta(1) - \theta(0) = \theta'(\widetilde{\lambda})$, for some $\widetilde{\lambda} \in (0,1)$. Note $\widetilde{z} = z_0 + \widetilde{\lambda}(z_1 - z_0)$

Compute

$$\theta'(\widetilde{\lambda}) = \frac{\int_t^{+\infty} d(T)\phi''\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)\left(\int_0^T (z_1(\tau_1) - z_0(\tau_1))g'(\widetilde{z}(\tau_1))d\tau_1\right)dT}{\int_t^{+\infty} d(T)\phi'\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)dT}$$

that can be written as

$$\theta'(\widetilde{\lambda}) = \frac{\int_t^{+\infty} d(T)\phi'\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)K_1(T)K_2(T)dT}{\int_t^{+\infty} d(T)\phi'\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)dT}$$

with

$$\begin{aligned} K_1(T) &= \frac{-\int_0^T (z_1(\tau) - z(\tau))g'(\widetilde{z}(\tau))d\tau}{\int_0^T g(\widetilde{z}(\tau))d\tau} \\ K_2(T) &= \left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)\frac{-\phi''\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)}{\phi'\left(\int_0^T g(\widetilde{z}(\tau))d\tau\right)} \end{aligned}$$

We have

$$K_2(\tau) \leq \sup_{\kappa\in[\omega u(c_m),\omega u(c_M)]} \frac{-\kappa\phi''(\kappa)}{\phi'(\kappa)}$$

Now use $g = u \circ [u']^{-1} \circ \exp$ to obtain that for any z

$$g'(z) = \frac{-u'(c_z)}{c_z u''(c_z)}\frac{c_z u'(c_z)}{u(c_z)}g(z)$$

[25] To prove that $\Delta \geq 0$ one can show that the function

$$f(x) = \left(\int_t^x k(\tau)g(\tau)d\tau\right)\left(\int_t^x h(\tau)g(\tau)d\tau\right) - \left(\int_t^x k(\tau)h(\tau)g(\tau)d\tau\right)\left(\int_t^x g(\tau)d\tau\right)$$

is non-decreasing (and therefore non-negative) for $x \geq t$.

with $c_z = [u']^{-1}(\exp(z))$.

It follows that

$$|K_1(T)| \leq \sup_{t\in[0,\omega]} ||z_1(t) - z_0(t)|| \left[\sup_{c\in[c_m,c_M]} \frac{-u'(c)}{cu''(c)}\right] \left[\sup_{c\in[c_m,c_M]} \frac{cu'(c)}{u(c)}\right]$$

Thus

$$\theta'(\widetilde{\lambda}) \leq \sup_{t\in[0,\omega]} ||z_1(t) - z_0(t)|| \left[\sup_{c\in[c_m,c_M]} \frac{-u'(c)}{cu''(c)}\right] \left[\sup_{c\in[c_m,c_M]} \frac{cu'(c)}{u(c)}\right] \left[\sup_{\kappa\in[\omega u(c_m),\omega u(c_M)]} \frac{-\kappa\phi''(\kappa)}{\phi'(\kappa)}\right]$$

which completes the proof of Lemma 1.

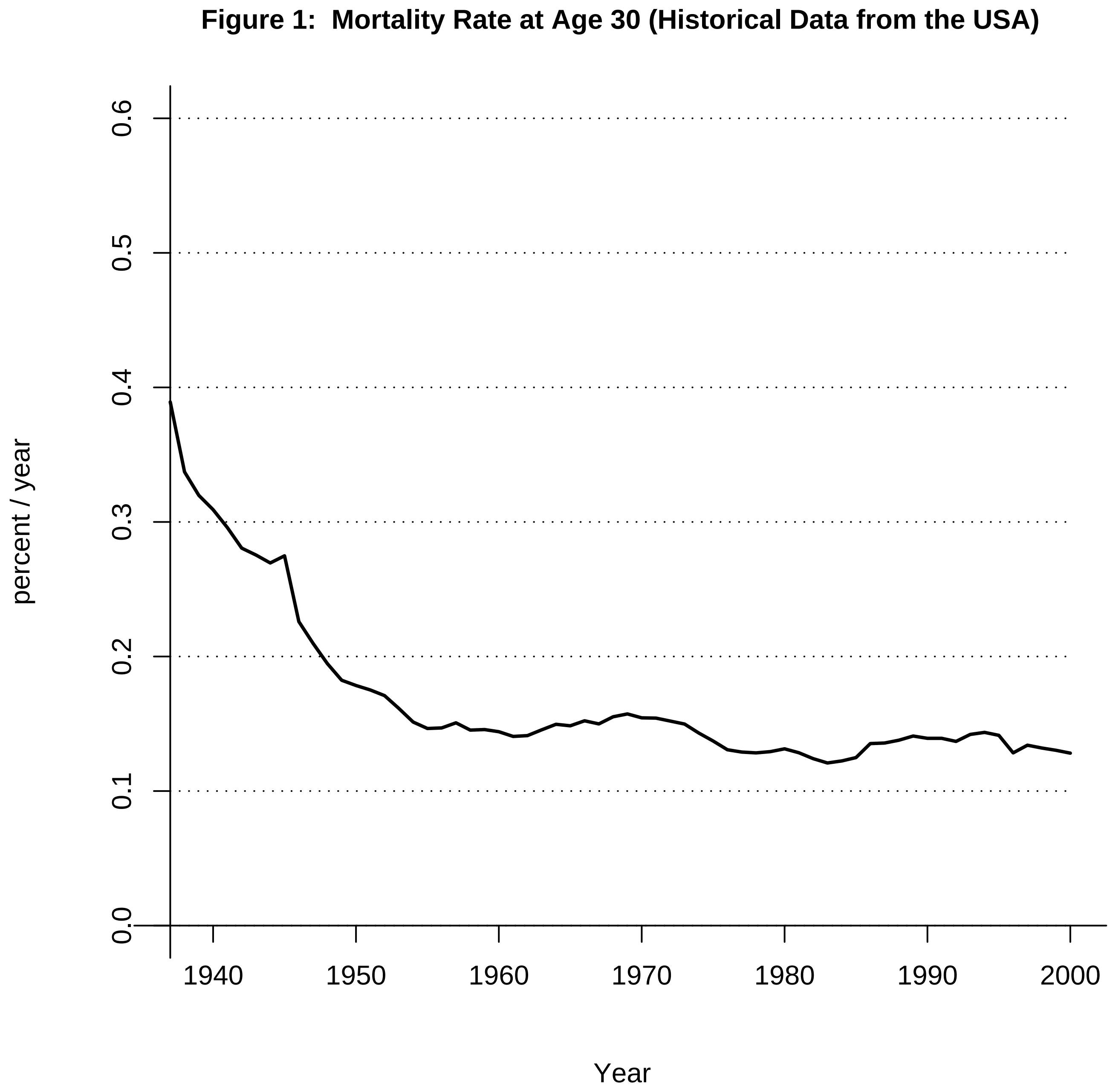
Figure 1: Mortality Rate at Age 30 (Historical Data from the USA)
percent / year
0.0
0.1
0.2
0.3
0.4
0.5
0.6
1940
1950
1960
1970
1980
1990
2000
Year

Figure 2: Life Expectancy at Age 30 (Historical Data from the USA)

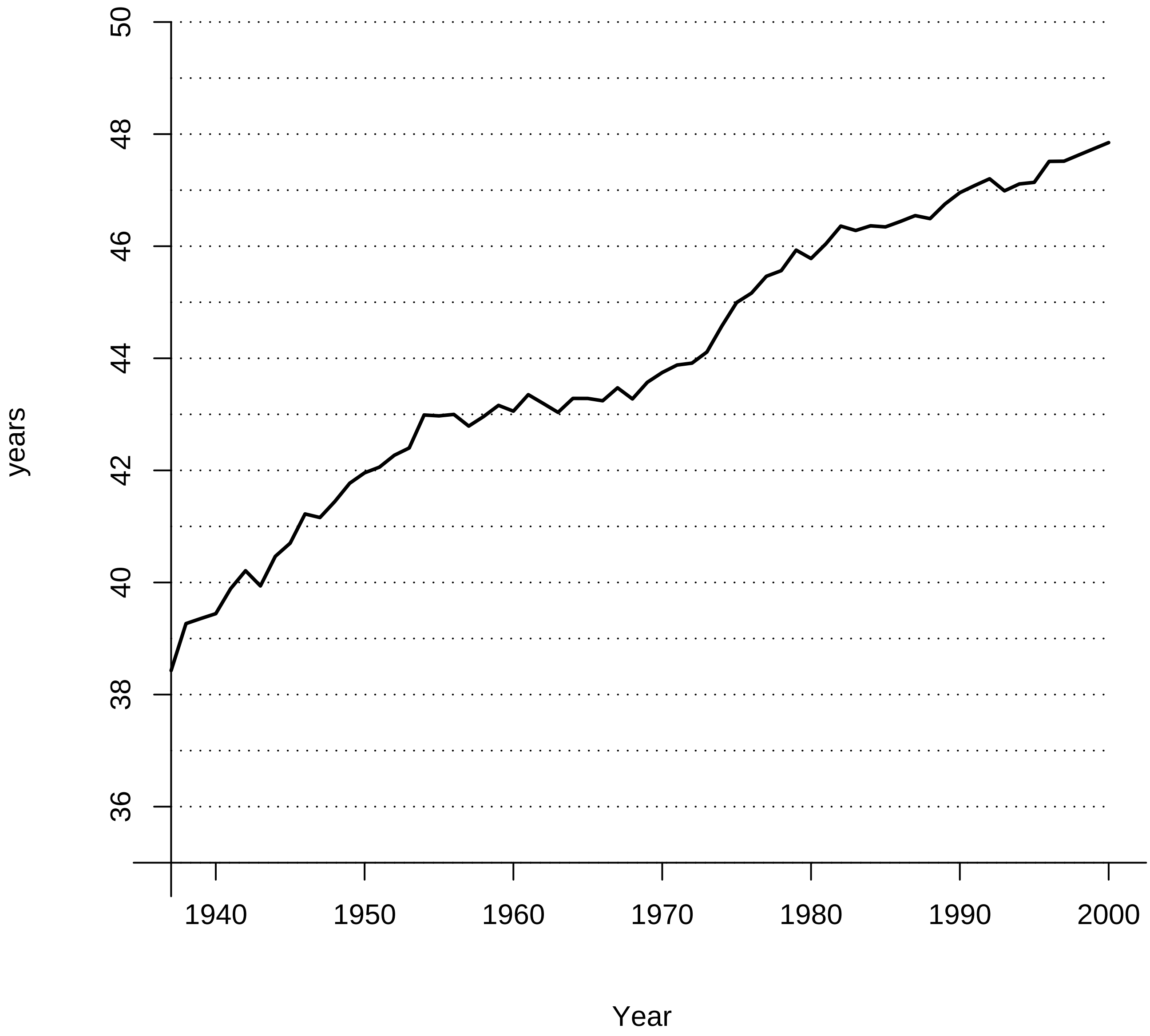

Figure 3: Rate of Discount at Age 30 According to Historical Mortality rates

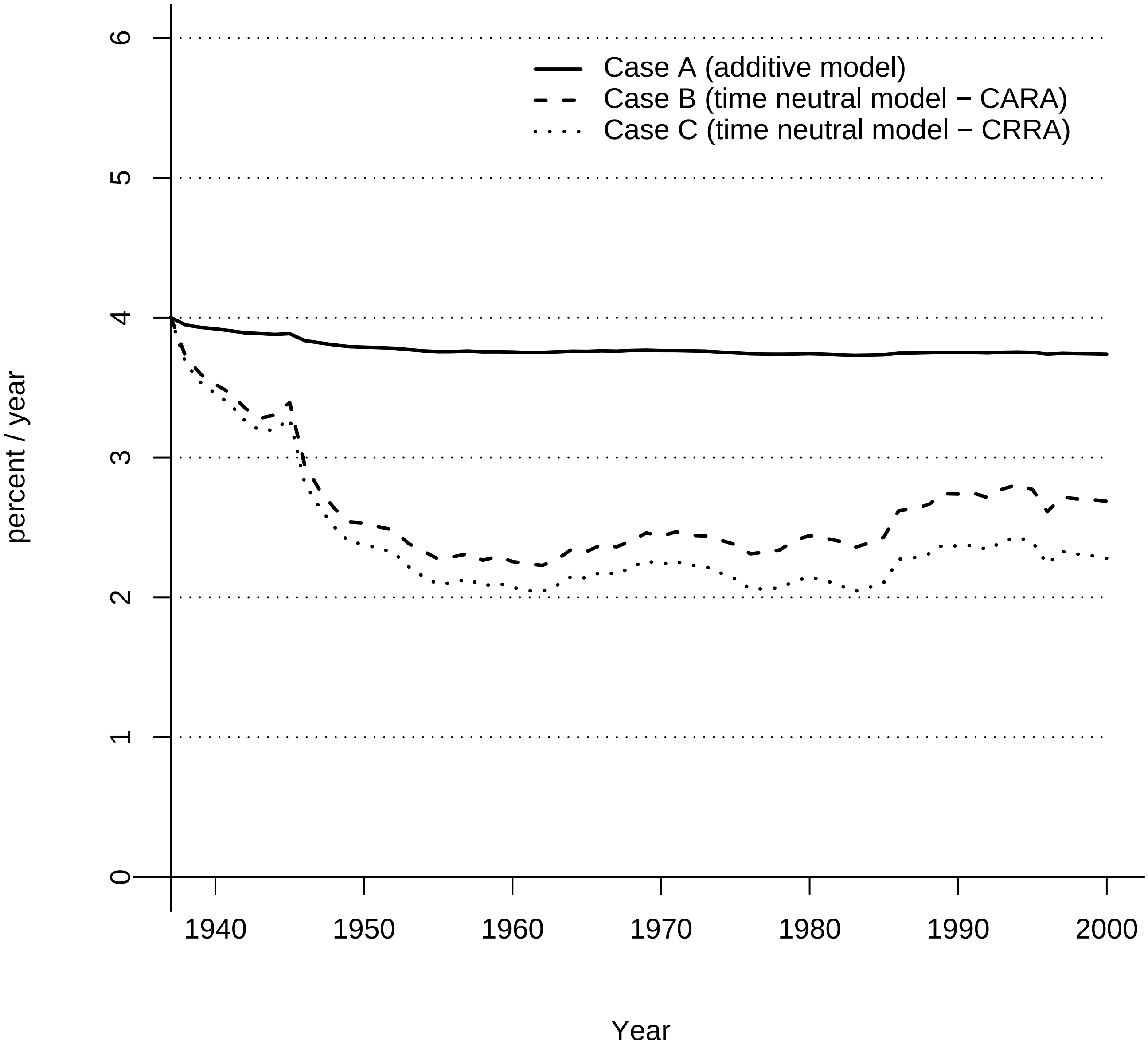

Figure 4

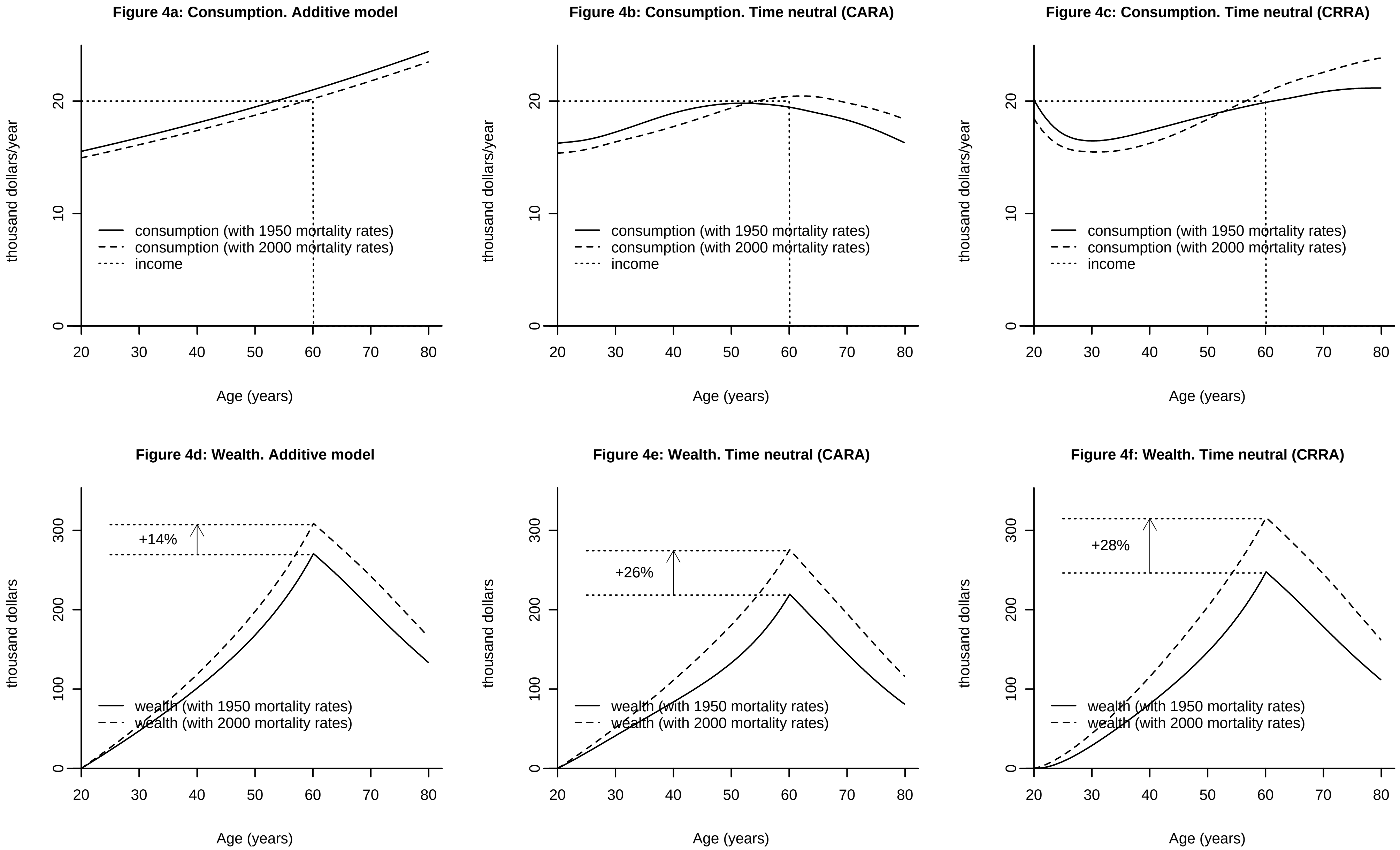
Figure 4a: Consumption. Additive model
Figure 4b: Consumption. Time neutral (CARA)
Figure 4c: Consumption. Time neutral (CRRA)
thousand dollars/year
Age (years)
consumption (with 1950 mortality rates)
consumption (with 2000 mortality rates)
income
Figure 4d: Wealth. Additive model
Figure 4e: Wealth. Time neutral (CARA)
Figure 4f: Wealth. Time neutral (CRRA)
thousand dollars
wealth (with 1950 mortality rates)
wealth (with 2000 mortality rates)
+14%
+26%
+28%